TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 12

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Homework 12.1 (High-dimensional Schwarz lemma*)**.** Let $f\colon B_1(0) \to \mathbf{C}$ be holomorphic such that $f(0) = 0$, where $B_1(0)$ denotes the unit ball in $\mathbf{C}^n$, where $n \geq 2$. Assume there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in B_1(0)$.

a. Show $|f(z)| \leq M\|z\|$ for every $z \in B_1(0)$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{C}^n$.
b. If $n = 1$, the equality $|f(z)| = M|z|$ for some nonzero $z \in B_1(0)$ implies $f(z) = Maz$ for some $a \in \partial B_1(0)$. In particular, when $M > 0$ then f is biholomorphic. Show that for $n \geq 2$ there is a holomorphic function $f\colon B_1(0) \to B_1(0)$ with $f(0) = 0$ and $\|f(z)\| = \|z\|$ for some $z \in B_1(0) \setminus \{0\}$ that is not even injective[1].

Homework 12.2 (A stronger version of the identity theorem)**.** Let $D \subset \mathbf{C}^n$ be a domain and $f\colon D \to \mathbf{C}$ be holomorphic. Assume there exists $a \in D$ such that for every multiindex $\alpha \in \mathbf{N}_0^n$, we have $\mathrm{D}^\alpha f(a) = 0$. Show $f = 0$.

Homework 12.3 (Consequences of Hartogs' extension theorem)**.** Let $n \geq 2$ and $U \subset \mathbf{C}^n$ be open. Show the following statements

a. If $f\colon U \setminus \{a\} \to \mathbf{C}$ is holomorphic for a given point $a \in \mathbf{C}^n$, then f can be extended to a holomorphic function $f\colon U \to \mathbf{C}$.
b. If $K \subset \mathbf{C}^n$ is compact and such that $\mathbf{C} \setminus K$ is connected, then every holomorphic function $f\colon \mathbf{C}^n \setminus K \to \mathbf{C}$ can be extended to an entire function.
c. If $f\colon U \to \mathbf{C}$ is holomorphic, then f cannot have an isolated zero.
d. If $f\colon \mathbf{C}^n \to \mathbf{C}$ is entire, then $\{f = 0\}$ is either empty or unbounded.

Homework 12.4 (Merry Christmas*)**.** Enjoy your holidays!

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[1] Cartan's uniqueness theorem states if $U \subset \mathbf{C}^n$ is bounded and $f\colon U \to U$ has a fixed point $a \in U$ with $\mathrm{D}f(a) = \mathrm{Id}$ then $f(z) = z$ for all $z \in U$.